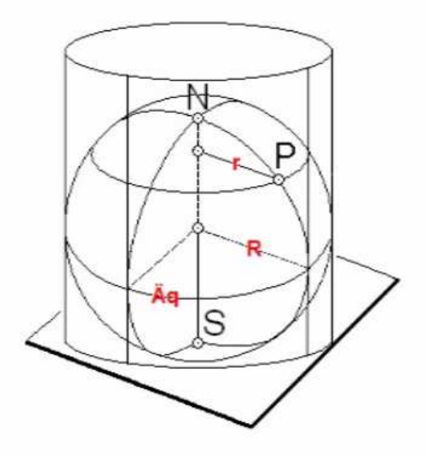
Friedrich Wilhelm Krücken

Are „quadratura circuli“ and Mercator's “loxodromy in 1569” related issues?

Prelude

In the years shortly before 1569 Gerhard Mercator finishes more than 20 years of intellectual reflection on the deficiencies of the quadratic flat map with the coded sentence - as it was a custom in those days - : “As we have considered this [I. e. the errors caused by the use of the flat map as a sea-chart], we have gradually increased the latitudes in the direction towards the two poles in proportion to the increase in width parallel beyond the level they have towards the equator”.



What is meant by "increase in width parallel in relation to the equator“?

Fig.1

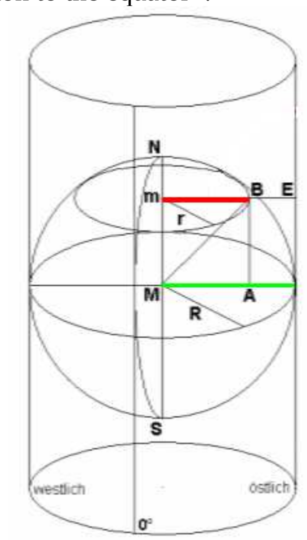
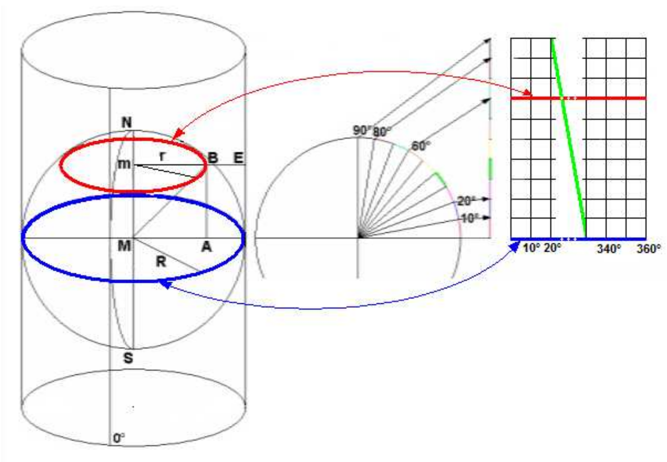


Fig.2

Well: Marinus (~ AD 100) increases the radii of the parallels of latitude: that is from r = mB to R = mE = MB.

The model of the 10° latitude distances is clearly visible:

Fig.3



a) the parallels of latitude become as long as the equator
b) but the distances among each other do not change.

Marinus enlarges all radii of the latitudes (for example mB) so that they finally

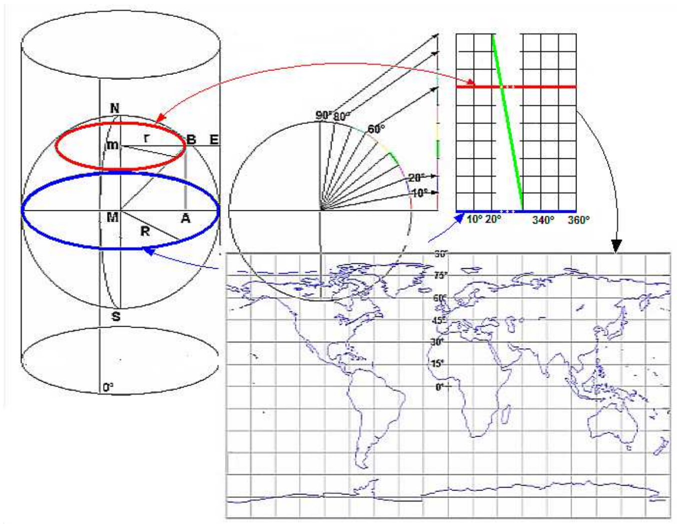


Fig.4

take the size of the equatorial radius R: MB = mE.

But Marinus leaves the latitude-distances – in the model the 10°-distances – in their original size.
In the end, the result is a map distorting everything in the north-south direction.
This type of a map is known as "equidistant”.

The deficiencies of the square flat map become now obvious:

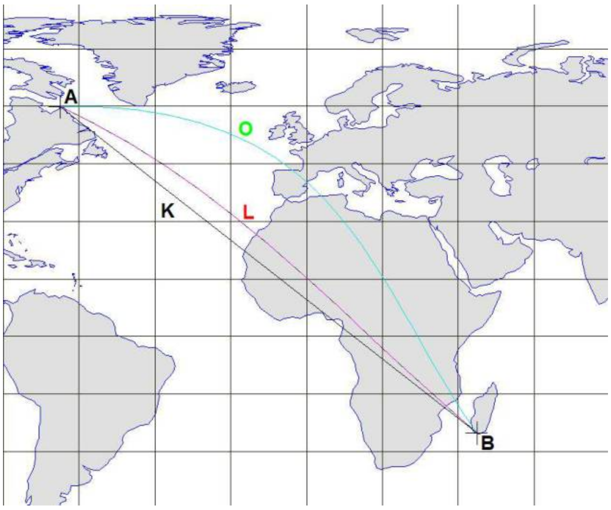


Fig.5

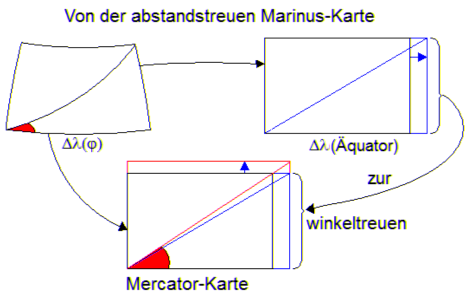
If you fly from A to B the course – marked by the green curve **O** – is the shortest one, because **O** is a great circle on the map as well as on the globe: and any great circle is the shortest connection between two points on the spherical earth.
Courses following great circles – except the lines of longitude or the equator – have the great disadvantage that the course angle must be reset from long to long.

On Mercator's globe of 1541, the red curve **L** appears as one of his double-curved course-lines which Willebrord of Royen – known as 'Snellius' - later called 'loxodrome' (crooked curve = rhumb) in contrast to 'orthodrome' (straight curve= great circle).

In 1533 Pedro Nunes, since 1529 Cosmograph of the King of Portugal, was the first to described the difference between orthodromic sailing and the loxodromic sailing, i. e. sailing along the constant compass-course. Gerhard Mercator was the first to realize and transfer Nunes' insights on his terrestrial globe (1541).

In those days, captains and first mates set their compass-course as straight lines, similar to the black curve **K**, because they believed these lines were the precise compass-courses.
In 1533 – after his return from the Rio de la Plata to Lisboa – Martin Alfonso de Sousa complained to Nunes: The king's hydrographers had provided him with false sea-charts for his voyage to the Rio de la Plata. Finally, he had missed the mouth of the Rio de la Plata by hundreds of Portuguese nautical miles to the north, although he had stuck precisely to the correct compass course. On the

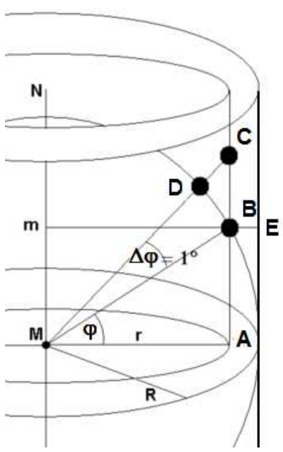
voyage back he hadn't either passed the equator at the very spot he had - correctly sailing the compass course - marked in his sea chart before.



And these are exactly the errors Gerhard Mercator tries to overcome with his map *Ad usum navigantium* now: Marinus' mistake was not to have enlarged the latitude distances towards the poles simultaneously with the latitude sizes.

Mercator writes:
"(Having considered it) ... we gradually increased the latitude (intervals of latitude) in the direction of both poles in proportion to the increase in width parallel (latitude sections) beyond the level they have towards the equator (equatorial section)".

To understand Mercator's words we have to determine the factor of enlargement first:
This isn't particularly difficult:



The ratio R : r = MB : MA gives this factor because it applies,

$$r \bullet R/r = R$$
or
$$mE = mB \bullet MB/MA = MB.$$

We remember
$$r = mB = MA$$
$$R = MB$$
therefore we favour as a proportional factor:

$$x = MB/MA.$$

Fig.6

As a result: You do not need any knowledge of trigonometry to determine the factor of similarity x.
Look at the figure 6 and think about how to be able to enlarge the arc BC and you will at once recognize the picture in perspective of the arc BD from M on the sectional cylinder belonging to r resp. φ.
We "simply" produce MD beyond D till it meets AB …: C. Therefore, the decisive question is:

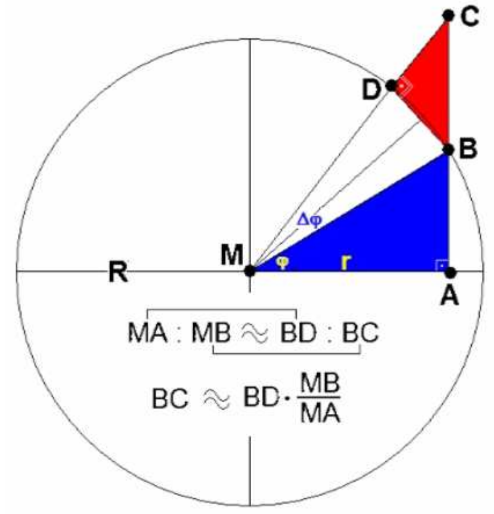
Is it possible to prove that BC is MB/MA times arc BD?

In order to start such a proof we go from the arc BD to the chord BD:

> Euclid’s theorem of similarity doesn't know triangles put together from straight lines and arcs.

And - unfortunately - we have to state that the angle CDB is not a right angle!

Here we are confronted with a fact Gerhard Mercator could describe to his neighbour, friend and (later) biographer, Walter Ghym as follows:



“My method of constructing a new map is nearly as difficult as trying to square the circle. Apparently, I can only carry out my constructions approximately just as it is possible to calculate the circumference of the circle resp. its area approximately as well. Unfortunately, I haven't a proof for this relation.”

Fig.7

And he draws the following conclusion – in mind –:

“But when I go on 'gradually' with the construction then the disadvantage will be insignificant:

If I choose the progression for each single degree so the angle CDB differs from a right angle by only half a degree. The chord BD differs from the arc BD by an order which I cannot state when drawing.”

> In fact. Even when R = 315 mm (1569) the difference is to small
> - it cant be drawn.

Let's take a closer look: The "almost" right-angled triangle BDC is “almost” similar to the right-angled triangle MAB.

Following *Euclid VI* 4 we can state: BC ≈ BD • MB/MA.
This means nothing but:
The same factor MB/MA which projects r on R, is the same as the Marinus BD-latitude-distance on the Mercator latitude-distance BC.

And BC will become a good approximation for the "true" distance!

Thus, Mercator's method has become reproducible in a high degree; BUT: Only one element is still missing.
To improve the "quality" of the construction we turn to Mercator's 'method of "simple" mid-latitude':

> Instead of projecting the Marinus-latitude intervals of 1° of the quadratic flat map onto the relevant Mercator-latitude distances on the new flat map "Ad usum navigantium" - and beginning from the equator – we project the Marinus-latitude of 0,5° - 1.5° on the Mercator-latitude of 0°-1°, the Marinus-latitude of 1.5° - 2.5° on the Mercator-latitude of 1° - 2° and so on … .

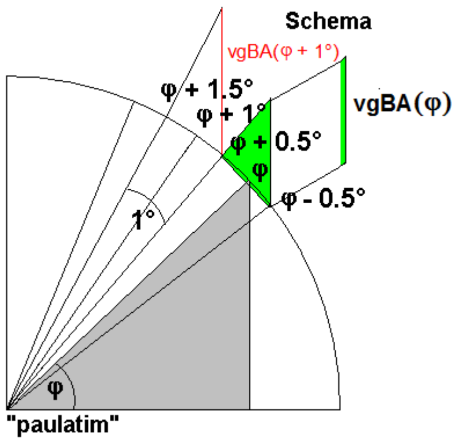


Fig.8

Thus, this reconstruction is solely based on the mathematical and cartographic knowledge Gerhard Mercator had:

- Euclid's theory of similarity
- Ptolemy's chord theory
- map construction for simple (or double) mid-latitudes.
- perspective

Any further knowledge of “higher” mathematics is not required.

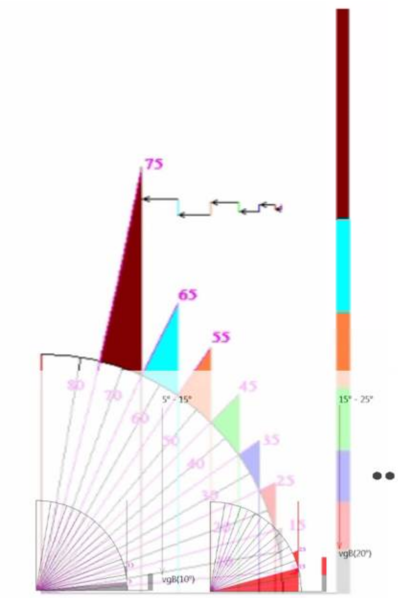
The birth of the loxodromy in 1569

Of course we don't know and – most probably – we'll never find out what happened in Gerhard Mercator's workshop when the increased latitudes of his map *Ad usum navigantium* were constructed with his (limited) knowledge of mathematics.

But the simpler and more compatible an elementary geometric-constructive suggestion is to the knowledge of Gerhard Mercator and within the limits of the drawing accuracy with Mercator's enlarged latitudes of 1569, the greater is the probability to have shown Mercator's idea.

The following sequence of pictures shows what might have happened on Mercator's drawing table after the constructions of the “prelude". The distances measured on the Basle copy are highly consistent with those constructed according to the “method of the sliding perspective” (cf. next pictures).

The model shows the reconstruction with 10° -distances.



To get on, only Gerhard Mercator's knowledge of mathematics are taken for granted

- perspective
- Euklid's theory of similarity (6th book)
- map-construcion following the mid-latitudes
- chord geometry of Ptolemy

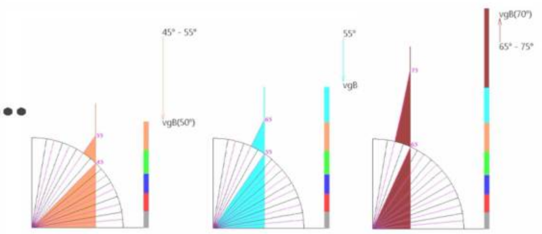


Fig.9 Fig.10

The mathematical reconstruction results from these conditions:

$$vgBA(\varphi) = \text{trunc}[2 \bullet \sin(0.5 \bullet (22/7)/180) \bullet R \bullet \sec((\varphi - 0.5) \bullet (22/7)/180) \bullet 10]/10$$

[„Mercator's formula”]

with

φ = 1, 2, … 79 , R = 315 mm | 125.112 mm
vgBA = increased intervals of latitude
vgB = increased latitude
$rad_{Mercator}$ = (22/7)/180
Fak11 = Basle copy = ISBN 978-3-00-035705-3

Theorie = theory = vgB(φ) =R•ln[tan(φ) + sec(φ)]
Rek = reconstruction of vgBA by using Mercator's formula:

$$vgBA(\varphi) = \mathrm{trunc}[2\bullet\sin(0.5\bullet rad_{Mercator})\bullet R\bullet\sec((\varphi-0.5)\bullet rad_{Mercator})\bullet 10]/10$$

Table 1

		I		II		III		IV		V		VI		VII
	Fak11		Theo-rie		Rek.		Fak11	* 2.518	Basel		Theorie		Rek.	
Grad	vgBA	vgB	vgBA	vgB	vgBA	vgB	vgBA	vgB	vgBA	vgB	vgBA	vgB	vgBA	vgB
1	2.1	2.1	2.18	2.18	2.1	2.1	5.3	5.3	5.5	5.5	5.498	5.498	5.5	5.5
5	2.4	10.8	2.19	10.93	2.1	10.5	6.0	27.2	5.6	27.5	5.515	27.524	5.5	27.5
10	2.1	21.2	2.22	21.95	2.2	21.3	5.3	53.4	5.5	55.2	5.574	55.259	5.5	55.0
15	2.1	32.1	2.27	31.14	2.2	32.3	5.3	80.8	5.8	83.5	5.679	83.425	5.6	82.9
20	2.2	43.2	2.33	44.59	2.3	43.5	5.5	108.8	5.8	112.4	5.832	112.259	5.8	111.5
25	2.3	54.9	2.42	56.41	2.4	55.1	5.8	138.2	6.0	141.4	6.042	142.026	6.0	141.0
30	2.5	66.9	2.53	68.72	2.5	67.2	6.3	168.5	6.2	171.9	6.317	173.031	6.3	171.8
35	2.6	79.0	2.68	81.68	2.6	79.9	6.5	198.9	6.8	203.8	6.671	205.644	6.6	204.1
40	2.7	92.0	2.87	95.45	2.8	93.4	6.8	231.7	6.9	237.9	7.125	240.317	7.1	238.6
45	3.1	106.6	3.12	110.27	3.0	108.0	7.8	268.4	7.6	274.2	7.708	277.633	7.7	275.7
50	3.2	122.1	3.43	126.45	3.3	123.9	8.1	307.4	8.2	314.0	8.466	318.365	8.4	316.2
55	3.6	139.0	3.86	144.41	3.7	141.6	9.1	350.0	9.4	358.1	9.468	363.584	9.4	361.2
60	4.1	158.3	4.43	164.77	4.3	161.7	13.3	398.6	10.7	409.3	10.833	414.842	10.8	412.3
65	5.3	181.4	5.27	188.48	5.0	185.2	13.3	456.8	12.4	467.8	12.772	474.533	12.7	481.8
70	5.8	209.1	6.54	217.12	6.2	213.6	14.6	526.5	16.0	539.7	15.702	546.656	15.7	543.8
75	8.4	245.4	8.72	253.68	8.1	249.9	21.2	617.9	20.7	632.4	20.580	638.691	20.6	635.7
79			11.99	292.81	10.9	288.9			27.8	732.5	27.593	737.226	27.6	734.3

If we calculate *Ad usum navigantium* [Basle copy = ISBN 978-3-00-035705-3]

(a) the vgBA and vgB for the *organum directorium* of the map:
R = 125.11 mm (Fak11), and
(b) for the map: R = 315 mm ('Basel'),

so we get the results above: Table 1 (see appendix II for further information).

And – more precisely – we get the following results:

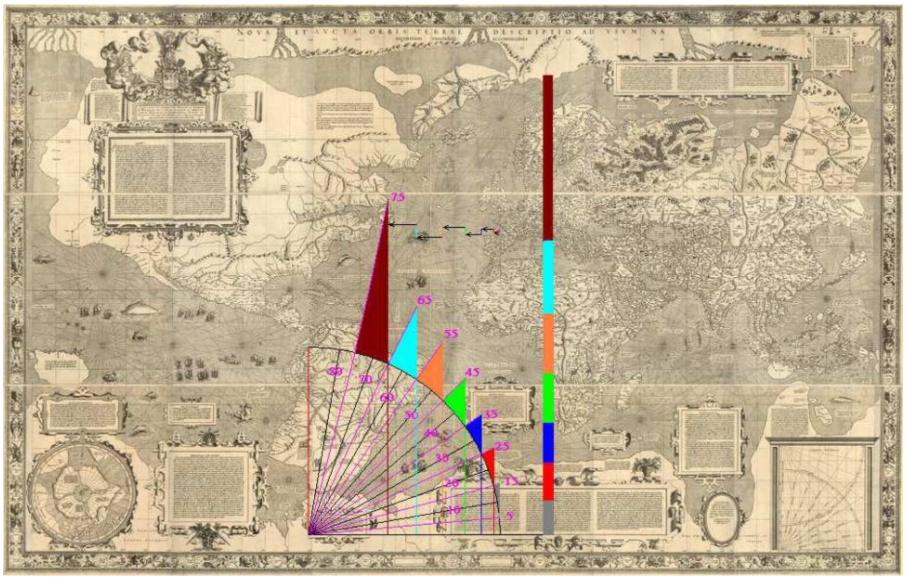


Fig. 11

(a) IV:V: The vgBA of the *organon* are not suitable for the construction of the vgBA of the map.

$$\rightarrow vgB_{map}(75°) - vgB_{organon}(75°) = 14.5 \text{ mm},$$

(b) V:VI Mercator's graphic construction (V) doesn't match the computing theory (VI). However, this doesn't justify its rejection:

(b1) The difference between the vgBA of the map and those of the theory amounts exactly from 1° to 79° 4.73 mm: $\Delta_{79°(map/theory)}$ = 4.726 mm ÷ 6.41‰.

(b2) The difference between the vgBA of the map and those of the reconstruction with "Mercator's formula" (in a literary, science-historical and science-theoretical context): 1.8 mm: $\Delta_{79°(map/reconstruction)}$ = 1.8 mm ÷ 2.44‰, let me say, in accordance with the barely concealed joy of Arthur Breusing:

> "To put straight: The deviations are so tiny that I [A. B.] was surprised by this fact. Anyway, the precision is sufficient for being applied by the seafarers, ad usum navigantium, and proofs - simultaneously - that Mercator, even if he did not mention it on the map particularly, that definitely recognized and applied that principle correctly." [1]

1 Arthur Breusing: Das Verebnen der Kugeloberfläche, 1892, 35 (my translation)

The differences between the vgBA measured in Basel (1992) on the original map and the vgBA reconstructed by “Mercator's formula” – being in the tenth-millimeter-area – verify the proximity of the reconstruction to the original:

$$\mu_{\text{Basel/reconstruction}} = -0.0228 \text{ mm} \quad \sigma_{\text{n-1, Basel/reconstruction}} = 0.2088 \text{ mm} \quad \rho_{\text{Basel/reconstruction}} = 0.9993.$$

On the other hand: the distance between the reconstruction and the theory is proved by

$$\mu_{\text{reconstruction/theory}} = -0.0370 \text{ mm} \quad \sigma_{\text{n-1,reconstruction/theory}} = 0.0316 \quad \rho_{\text{reconstruction/theory}} = 1.000\text{[2]}.$$

For the vgBA-distances Basle : theory is valid

$$\mu_{\text{Basel/theory}} = -0.0598 \text{ mm} \quad \sigma_{\text{n-1,Basel/theory}} = 0.2191 \text{ mm.[3]}$$

Therefore these are the results: The criticisms in 2005[4], 2012[5] and 2014[6] are to be rejected in entirely.

Quadratura circuli
or: what does loxodromy have to do with the squaring of the circle?

There is every indication that Gerhard Mercator's method of the enlarged latitudes – the reconstruction of which we have developed just before – corresponds with the method Gerhard Mercator used, most probably.
This applies the more as we succeeded as early as 1996 not only to establish a connection between Mercator's construction of the enlarged latitudes and the problem of “squaring the circle” as Mercator himself called it.[7]

Repeatedly, Gerhard Mercator had emphasized in talks with his friend, neighbour and (later) biographer, Walter Ghym, that his method of unrolling the sphere equiangularly onto the cylinder corresponds to the squaring of the circle, nothing but the (final) evidence seemed to be missing (“… as I – Walter Ghym – have heard from him several times.”)

2 Of course!
3 Attune my supposition, that Gerhard Mercator took as a basis the radius of a renish foot (313.85 mm), follows: $\mu_{\text{Basel/reconstruction}} = -0.0127$ mm $\sigma_{\text{n-1, Basel/reconstruction}} = 0.2255$ mm. However, it is valid in every case: the deviations – on average – have hardly the size of the line-widths (0.2 mm) of the map.
4 Raymond D'Hollander: *Loxodromie et le projection de Mercator*
5 Joaquim Alves Gaspar: *Squaring the circle; how Mercator did it in 1569* (St. Niklaas [Mercator Revisited]) – now http://lisboa.academia.edu/JoaquimGaspar (January 2013)
6 Joaquim Alves Gaspar & Henrique Leitão: *Squaring the Circle: How Mercator Constructed His Projection in 1569*, IMAGO MUNDI 66:1, 1-24
7 Text published in 1998 – *Praxis der Mathematik* | 2009 – *Ad maiorem Gerardi Mercatoris gloriam* I

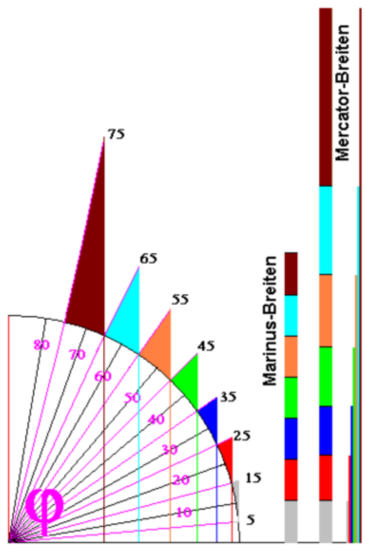
> Sic quadraturae circuli **respondet**, ut nihil deesse videatur, praeterquam quod demonstratione careat [ut ex illius ore aliquoties audivi].

The evidence developed by me – the *demonstratio* – is (only!) based on the mathematical knowledge, Mercator had:

> (a) similarity and perspective and in the transition from arithmetic to geometry:
> (b) fractions, i. e. only the arithmetic of rational numbers. Irrationalities like roots and any other knowledge of "higher" mathematics are not necessary.

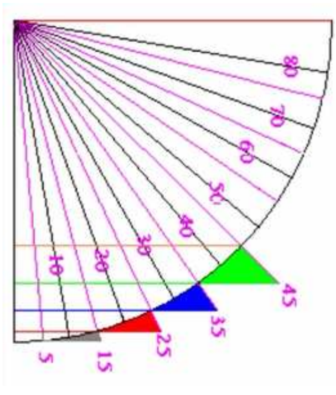
(1) Let's start with the "loxodromic" triangles …

Fig. 12



(2) … and prepare the diagram for further processing: From now on we consider only the eighth part of the full circle, and go on from the "sliding" perspective to the "simple" one:

Fig. 13



(3) However, in order to be able to proceed from geometry to arithmetic, we leave behind the uniform division of the circle and go on to a uniform division

of the tangent ST into n equal parts.

Fig. 15 ← Fig. 14

Fig. 15

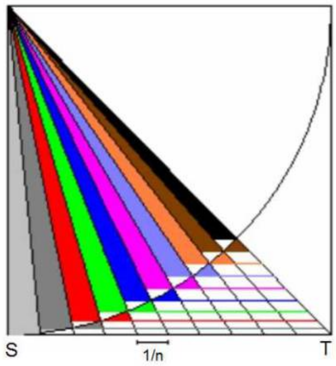
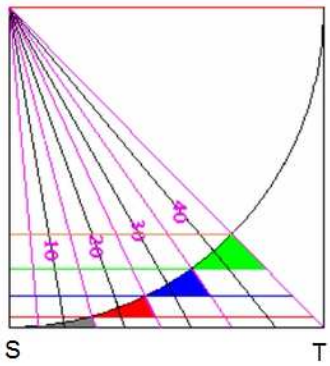


Fig. 14



(4) Then we turn our attention to the triangles which (still incomplete) exhaust resp. circumscribe the eighth-circle:

We choose the i^{th} triangle MCF as a partial figure and draw the heights ***h***, ***H***.

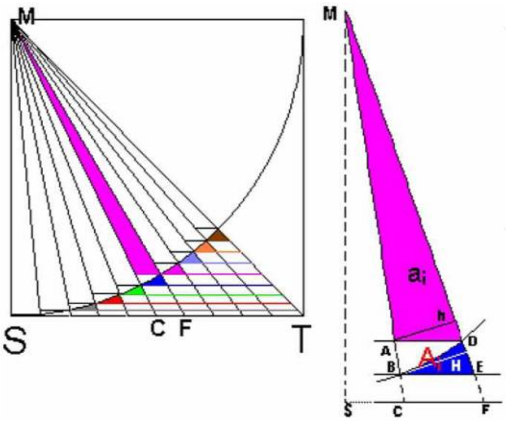


Fig. 16

(5) The similarity of the triangles MAD (surface $\mathbf{a_i}$*) and MBE (surface* $\mathbf{A_i}$*) finally leads to the following equation:*

$$a_i : A_i = MD^2 : ME^2$$

(6) You see immediately that

$$a_6 = A_7, \ldots, a_i = A_{i+1}$$

followed by the definitions

$a :=: a_1 + a_2 + \ldots + a_n$ and
$A :=: A_1 + A_2 + \ldots + A_n \rightarrow$
$A - a = A_1 - a_n$

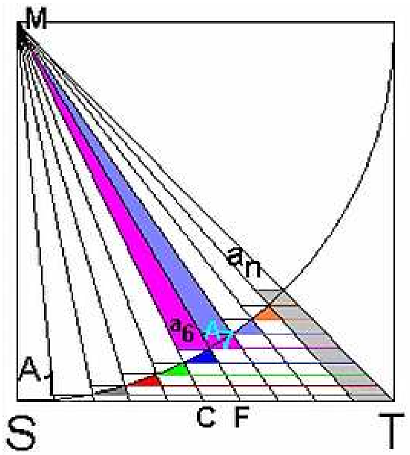


Fig. 17

(7) This means:
the grey coloured trapezoid in the n^{th} triangle illustrates the difference between the inner triangles, exhausting the eighth-circle and the circumscribed triangles.

An increase in the numbers n means – you see – a decrease of the last, the n^{th} triangle:

> Does n become as big as you like, the area of the grey trapezoid will become as small as you like: internal exhausting and external enclosure approach each other freely.

But: mathematics are not prepared to accept qualitative statements, maths demand reasonable, arithmetically formulated statements. Here, we inform about the result – without straining anybody here too much[8]:

8 Further information: Friedrich Wilhelm Krücken: *Ad maiorem Gerardi Mercatoris gloriam*, vol. I, Münster 2009, 168-170

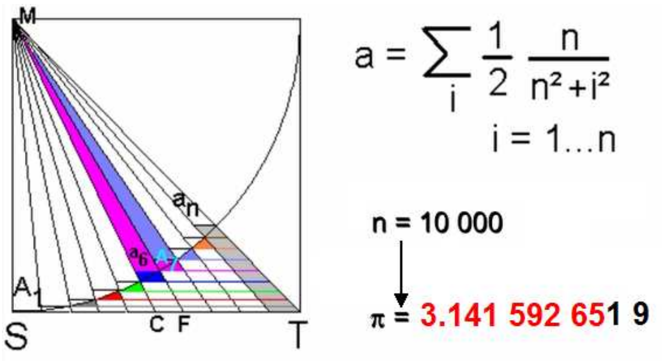


Fig.18

*If we choose n = 10 000 – and the mean between a and **A** –, the result will be π = 3.141,592,651,9 eighth decimal places exactly.*

An astonishing result: Gerhard Mercator's attempt at the squaring the circle ends in such an impressive result – solely with the help of fractions, the arithmetical form of the theory of similarity (and a little bit of series calculation).

In the end we have to emphasize again (and keep) that the figure – derived from loxodromics precisely – exactly hits the idea of Mercator's “respondet” in the text:

> [Brevissimo temporis curriculo intercedente …, proposuit nova et convenientissima sphaeram in plano extendendo, quae]
> Sic quadraturae circuli **respondet**, ut nihil deesse videatur, praeterquam quod demonstratione careat [ut ex illius ore aliquoties audivi].
>
> respondēre =
> ... correspond, agree with, to be similar
> (Livius. Cicero: Georges[9] B a).

However a text like the following:

> "According to the testimony of Walter Ghim, Mercator said more than once that his invention would **answer** to the squaring of the circle, so perfectly that nothing seemed to lacking save a proof."[10]

9 Karl Ernst Georges: *Lateinisch-deutsches Wörterbuch*, 11. A., Basel 1962
10 J. A. Gaspar: *Squaring the circle* … (2012) p. 1

misses completely the sense of Gerhard Mercator's remark as such a statement – firstly – ignores the context of Mercator's opinion completely and – secondly – follows an incorrect translation from Latin of the year 2000.

A table of rhumbs[11] neither couldn't nor can't **answer** the question about squaring the circle – and was fully aware of this problem. But the possibility which Mercator had found, namely to be able to connect his "inventio" of 1569 with the "quadratura circuli" by means of a "simple" perspective, only waited for a proof – a "demonstratio".

The explanation why Gerhard Mercator had not found this proof finds its simple reason in the fact that he was not yet able to realize the necessary shift of paradigm – the transition from geometry to arithmetic.[12]

The outline of the loxodromic structure of the map of 1569, accompanied by the speculation about its relation to the squaring of the circle, shows undoubtedly

> a) that the **self-taught** mathematician Gerhard Mercator had well understood his doings – in 1569,
> b) that the ancient topic of squaring the circle was represented correctly and brilliantly in its methodical relationship to the loxodromic theory of cartography by Gerhard Mercator and therefore
> c) he must find his place not only in the history of cartography but in the history of mathematics, too – practically as well as theoretically.

AD MAIOREM GERARDI MERCATORIS GLORIAM

11 J. A. Gaspar & H. Leitão: *Squaring the Circle …* (2014), 10-13, 17, note 55. Also (now: June 2014) in: H. Leitão & J. A. Gaspar: *Globes, Rhumb Tables, and the Pre-History of the Mercator Projection* (L/G:) IMAGO MUNDI 66:2, 180-195, here: 190f.

12 Though he knew first approaches to an elementary series calculation (Michael Stifel, 1544) – they didn't challenge him.

Appendix I

Table 1: Reconstructions
BASEL vgBA = 1992 measured latitudinal-distances (northern)
Reconstruction ← R = 315 mm
1rhFuss ← R = 313.85 mm
.br = Basel/Rek.315 entsprechend: Mittelwerte und Varianzen
.br_rhF = Basel/Rek.313.85
.rt = Rek.315/Theorie
.bt = Basel/Theorie
Theory : vgB(φ) = ln[tan+sec](φ) = Meridian parts$_{R=1}$ → Meridional parts = Meridian parts$_{R=1}$/arc 1°

BASEL 1992 und Rekonstruktion[en]

Grad	BASEL vgBA	BASEL vgB	Rekonstruktion vgBA	Rekonstruktion vgB	1rhFuss vgBA	1rhFuss vgB	Differenzen der vgBA dbr	dbr_rhF	drt	dbt
1	5.5	5.5	5.5	5.5	5.4	5.4	0.0	0.1	0.00	0.00
2	5.4	10.9	5.5	11.0	5.4	10.8	-0.1	0.0	0.00	-0.10
3	5.4	16.3	5.5	16.5	5.4	16.2	-0.1	0.0	-0.00	-0.10
4	5.6	21.9	5.5	22.0	5.4	21.6	0.1	0.2	-0.01	0.09
5	5.6	27.5	5.5	27.5	5.4	27.0	0.1	0.2	-0.01	0.09
6	5.6	33.1	5.5	33.0	5.5	32.5	0.1	0.1	-0.02	0.08
7	5.6	38.7	5.5	38.5	5.5	38.0	0.1	0.1	-0.03	0.07
8	5.5	44.2	5.5	44.0	5.5	43.5	0.0	0.0	-0.05	-0.05
9	5.5	49.7	5.5	49.5	5.5	49.0	0.0	0.0	-0.06	-0.06
10	5.5	55.2	5.5	55.0	5.5	54.5	0.0	0.0	-0.07	-0.07
11	5.5	60.7	5.5	60.5	5.5	60.0	0.0	0.0	-0.09	-0.09
12	5.7	66.4	5.6	66.1	5.5	65.5	0.1	0.2	-0.01	0.09
13	5.6	72.0	5.6	71.7	5.6	71.1	0.0	0.0	-0.03	-0.03
14	5.7	77.7	5.6	77.3	5.6	76.7	0.1	0.1	-0.05	0.05
15	5.8	83.5	5.6	82.9	5.6	82.3	0.2	0.2	-0.08	0.12
16	5.7	89.2	5.7	88.6	5.6	87.9	0.0	0.1	-0.01	-0.01
17	5.9	95.1	5.7	94.3	5.7	93.6	0.2	0.2	-0.03	0.17
18	5.7	100.8	5.7	100.0	5.7	99.3	0.0	0.0	-0.06	-0.06
19	5.8	106.6	5.7	105.7	5.7	105.0	0.1	0.1	-0.10	0.00
20	5.8	112.4	5.8	111.5	5.8	110.8	0.0	0.0	-0.03	-0.03
21	5.6	118.0	5.8	117.3	5.8	116.6	-0.2	-0.2	-0.07	-0.27
22	5.7	123.7	5.9	123.2	5.8	122.4	-0.2	-0.1	-0.01	-0.21
23	5.8	129.5	5.9	129.1	5.9	128.3	-0.1	-0.1	-0.05	-0.15
24	5.9	135.4	5.9	135.0	5.9	134.2	0.0	0.0	-0.10	-0.10
25	6.0	141.4	6.0	141.0	6.0	140.2	0.0	0.0	-0.04	-0.04
26	5.6	147.0	6.0	147.0	6.0	146.2	-0.4	-0.4	-0.09	-0.49
27	6.3	153.3	6.1	153.1	6.1	152.3	0.2	0.2	-0.04	0.16
28	6.4	159.7	6.2	159.3	6.1	158.4	0.2	0.3	0.00	0.20
29	6.0	165.7	6.2	165.5	6.2	164.6	-0.2	-0.2	-0.06	-0.26
30	6.2	171.9	6.3	171.8	6.2	170.8	-0.1	0.0	-0.02	-0.12
31	5.7	177.6	6.3	178.1	6.3	177.1	-0.6	-0.6	-0.08	-0.68
32	6.0	183.6	6.4	184.5	6.4	183.5	-0.4	-0.4	-0.05	-0.45
33	6.6	190.2	6.5	191.0	6.4	189.9	0.1	0.2	-0.02	0.08
34	6.8	197.0	6.5	197.5	6.5	196.4	0.3	0.3	-0.09	0.21
35	6.8	203.8	6.6	204.1	6.6	203.0	0.2	0.2	-0.07	0.13
36	6.6	210.4	6.7	210.8	6.7	209.7	-0.1	-0.1	-0.05	-0.15
37	6.7	217.1	6.8	217.6	6.8	216.5	-0.1	-0.1	-0.04	-0.14
38	6.9	224.0	6.9	224.5	6.9	223.4	0.0	0.0	-0.03	-0.03
39	7.0	231.0	7.0	231.5	7.0	230.4	0.0	0.0	-0.03	-0.03
40	6.9	237.9	7.1	238.6	7.1	237.5	-0.2	-0.2	-0.03	-0.23

Grad	BASEL vgBA	BASEL vgB	Rekonstruktion vgBA	Rekonstruktion vgB	1rhFuss vgBA	1rhFuss vgB	Differenzen der vgBA dbr	dbr_rhF	drt	dbt
41	6.8	244.7	7.2	245.8	7.2	244.7	-0.4	-0.4	-0.03	-0.43
42	7.2	251.9	7.3	253.1	7.3	252.0	-0.1	-0.1	-0.04	-0.14
43	7.2	259.1	7.4	260.5	7.4	259.4	-0.2	-0.2	-0.06	-0.26
44	7.5	266.6	7.5	268.0	7.5	266.9	0.0	0.0	-0.08	-0.08
45	7.6	274.2	7.7	275.7	7.6	274.5	-0.1	0.0	-0.01	-0.11
46	7.8	282.0	7.8	283.5	7.8	282.3	0.0	0.0	-0.04	-0.04
47	7.8	289.8	7.9	291.4	7.9	290.2	-0.1	-0.1	-0.09	-0.19
48	8.0	297.8	8.1	299.5	8.1	298.3	-0.1	-0.1	-0.04	-0.14
49	8.0	305.8	8.3	307.8	8.2	306.5	-0.3	-0.2	0.00	-0.30
50	8.2	314.0	8.4	316.2	8.4	314.9	-0.2	-0.2	-0.07	-0.27
51	8.2	322.2	8.6	324.8	8.6	323.5	-0.4	-0.4	-0.04	-0.44
52	8.6	330.8	8.8	333.6	8.8	332.3	-0.2	-0.2	-0.03	-0.23
53	8.9	339.7	9.0	342.6	9.0	341.3	-0.1	-0.1	-0.03	-0.13
54	9.0	348.7	9.2	351.8	9.2	350.5	-0.2	-0.2	-0.04	-0.24
55	9.4	358.1	9.4	361.2	9.4	359.9	0.0	0.0	-0.07	-0.07
56	9.7	367.8	9.7	370.9	9.6	369.5	0.0	0.1	-0.01	-0.01
57	9.9	377.7	9.9	380.8	9.9	379.4	0.0	0.0	-0.06	-0.06
58	10.4	388.1	10.2	391.0	10.2	389.6	0.2	0.2	-0.03	0.17
59	10.5	398.6	10.5	401.5	10.4	400.0	0.0	0.1	-0.02	-0.02
60	10.7	409.3	10.8	412.3	10.8	410.8	-0.1	-0.1	-0.03	-0.13
61	11.0	420.3	11.1	423.4	11.1	421.9	-0.1	-0.1	-0.07	-0.17
62	11.4	431.7	11.5	434.9	11.4	433.3	-0.1	0.0	-0.02	-0.12
63	11.7	443.4	11.9	446.8	11.8	445.1	-0.2	-0.1	-0.01	-0.21
64	12.0	455.4	12.3	459.1	12.2	457.3	-0.3	-0.2	-0.02	-0.32
65	12.4	467.8	12.7	471.8	12.7	470.0	-0.3	-0.3	-0.07	-0.37
66	13.0	480.8	13.2	485.0	13.2	483.2	-0.2	-0.2	-0.06	-0.26
67	13.9	494.7	13.8	498.8	13.7	496.9	0.1	0.2	0.01	0.11
68	14.1	508.8	14.3	513.1	14.3	511.2	-0.2	-0.2	-0.07	-0.27
69	14.9	523.7	15.0	528.1	14.9	526.1	-0.1	0.0	-0.00	-0.10
70	16.0	539.7	15.7	543.8	15.6	541.7	0.3	0.4	-0.00	0.30
71	16.5	556.2	16.4	560.2	16.4	558.1	0.1	0.1	-0.07	0.03
72	17.2	573.4	17.3	577.5	17.2	575.3	-0.1	0.0	-0.03	-0.13
73	18.7	592.1	18.3	595.8	18.2	593.5	0.4	0.5	0.01	0.41
74	19.6	611.7	19.3	615.1	19.3	612.8	0.3	0.3	-0.06	0.24
75	20.7	632.4	20.6	635.7	20.5	633.3	0.1	0.2	0.02	0.12
76	22.5	654.9	22.0	657.7	21.9	655.2	0.5	0.6	0.03	0.53
77	23.7	678.6	23.6	681.3	23.5	678.7	0.1	0.2	0.04	0.14
78	26.1	704.7	25.4	706.7	25.3	704.0	0.7	0.8	-0.01	0.69
79	27.8	732.5	27.6	734.3	27.5	731.5	0.2	0.3	0.01	0.21

Mittelwert B:R = -0.0228 Varianz = 0.2088
Mittelwert R:T = -0.0370 Varianz = 0.0316
Mittelwert B:T = -0.0598 Varianz = 0.2191
Mittelwert B:RrhF = 0.0127 Varianz = 0.2255 [alles in mm]

Appendix II

Table 2: organon : map : reconstruction : theory

	I		II		III		IV		V		VI		VII	
	Fak11		Theorie		Rekonstruktion		Fak11	×2.518	Basel		Rekonstruktion		Theorie	
1	2.1	2.1	2.18	2.18	2.1	2.1	5.3	5.3	5.5	5.5	5.5	5.5	5.498	5.498
2	2.0	4.1	2.19	4.37	2.1	4.2	5.0	10.3	5.4	10.9	5.5	11.0	5.500	10.998
3	2.1	6.2	2.19	6.55	2.1	6.3	5.3	15.6	5.4	16.3	5.5	16.5	5.503	16.501
4	2.2	8.4	2.19	8.74	2.1	8.4	5.5	21.2	5.6	21.9	5.5	22.0	5.508	22.009
5	2.4	10.8	2.19	10.93	2.1	10.5	6.0	27.2	5.6	27.5	5.5	27.5	5.515	27.524
6	1.9	12.7	2.20	13.13	2.1	12.6	4.8	32.0	5.6	33.1	5.5	33.0	5.523	33.047
7	2.1	14.8	2.20	15.32	2.1	14.7	5.3	37.3	5.6	38.7	5.5	38.5	5.533	38.581
8	2.2	17.0	2.21	17.53	2.2	16.9	5.5	42.8	5.5	44.2	5.5	44.0	5.545	44.126
9	2.1	19.1	2.21	19.73	2.2	19.1	5.3	48.1	5.5	49.7	5.5	49.5	5.559	49.685
10	2.1	21.2	2.22	21.95	2.2	21.3	5.3	53.4	5.5	55.2	5.5	55.0	5.574	55.259
11	2.2	23.4	2.23	24.17	2.2	23.5	5.5	58.9	5.5	60.7	5.5	60.5	5.591	60.851
12	2.2	25.6	2.24	26.40	2.2	25.7	5.5	64.5	5.7	66.4	5.6	66.1	5.610	66.461
13	2.2	27.8	2.25	28.63	2.2	27.9	5.5	70.0	5.6	72.0	5.6	71.7	5.631	72.092
14	2.2	30.0	2.26	30.88	2.2	30.1	5.5	75.5	5.7	77.7	5.6	77.3	5.654	77.747
15	2.1	32.1	2.27	33.14	2.2	32.3	5.3	80.8	5.8	83.5	5.6	82.9	5.679	83.425
16	2.2	34.3	2.28	35.40	2.2	34.5	5.5	86.4	5.7	89.2	5.7	88.6	5.705	89.131
17	2.2	36.5	2.29	37.68	2.2	36.7	5.5	91.9	5.9	95.1	5.7	94.3	5.734	94.865
18	2.2	38.7	2.30	39.97	2.2	38.9	5.5	97.4	5.7	100.8	5.7	100.0	5.765	100.629
19	2.3	41.0	2.32	42.27	2.3	41.2	5.8	103.2	5.8	106.6	5.7	105.7	5.797	106.427
20	2.2	43.2	2.33	44.59	2.3	43.5	5.5	108.8	5.8	112.4	5.8	111.5	5.832	112.259
21	2.2	45.4	2.35	46.92	2.3	45.8	5.5	114.3	5.6	118.0	5.8	117.3	5.870	118.129
22	2.3	47.7	2.36	49.27	2.3	48.1	5.8	120.1	5.7	123.7	5.9	123.2	5.909	124.038
23	2.4	50.1	2.38	51.63	2.3	50.4	6.0	126.2	5.8	129.5	5.9	129.1	5.951	129.989
24	2.5	52.6	2.40	54.01	2.3	52.7	6.3	132.4	5.9	135.4	5.9	135.0	5.995	135.984
25	2.3	54.9	2.42	56.41	2.4	55.1	5.8	138.2	6.0	141.4	6.0	141.0	6.042	142.026
26	2.2	57.1	2.44	58.83	2.4	57.5	5.5	143.8	5.6	147.0	6.0	147.0	6.091	148.117
27	2.2	59.3	2.46	61.27	2.4	59.9	5.5	149.3	6.3	153.3	6.1	153.1	6.143	154.260
28	2.5	61.8	2.48	63.73	2.4	62.3	6.3	155.6	6.4	159.7	6.2	159.3	6.198	160.459
29	2.6	64.4	2.51	66.22	2.4	64.7	6.5	162.2	6.0	165.7	6.2	165.5	6.256	166.715
30	2.5	66.9	2.53	68.72	2.5	67.2	6.3	168.5	6.2	171.9	6.3	171.8	6.317	173.031
31	2.3	69.2	2.56	71.26	2.5	69.7	5.8	174.2	5.7	177.6	6.3	178.1	6.381	179.412
32	2.3	71.5	2.59	73.82	2.5	72.2	5.8	180.0	6.0	183.6	6.4	184.5	6.448	185.860
33	2.5	74.0	2.62	76.41	2.5	74.7	6.3	186.3	6.6	190.2	6.5	191.0	6.519	192.379
34	2.4	76.4	2.65	79.03	2.6	77.3	6.0	192.4	6.8	197.0	6.5	197.5	6.593	198.972
35	2.6	79.0	2.68	81.68	2.6	79.9	6.5	198.9	6.8	203.8	6.6	204.1	6.671	205.644
36	2.5	81.5	2.72	84.36	2.6	82.5	6.3	205.2	6.6	210.4	6.7	210.8	6.753	212.397
37	2.5	84.0	2.75	87.08	2.7	85.2	6.3	211.5	6.7	217.1	6.8	217.6	6.839	219.236
38	2.6	86.6	2.79	89.83	2.7	87.9	6.5	218.1	6.9	224.0	6.9	224.5	6.930	226.166
39	2.7	89.3	2.83	92.62	2.7	90.6	6.8	224.9	7.0	231.0	7.0	231.5	7.025	233.191
40	2.7	92.0	2.87	95.45	2.8	93.4	6.8	231.7	6.9	237.9	7.1	238.6	7.125	240.317
41	2.7	94.7	2.92	98.32	2.8	96.2	6.8	238.5	6.8	244.7	7.2	245.8	7.230	247.547
42	2.7	97.4	2.96	101.24	2.9	99.1	6.8	245.3	7.2	251.9	7.3	253.1	7.341	254.888
43	3.0	100.4	3.01	104.20	2.9	102.0	7.6	252.8	7.2	259.1	7.4	260.5	7.457	262.345
44	3.1	103.5	3.06	107.21	3.0	105.0	7.8	260.6	7.5	266.6	7.5	268.0	7.580	269.924
45	3.1	106.6	3.12	110.27	3.0	108.0	7.8	268.4	7.6	274.2	7.7	275.7	7.708	277.633
46	3.0	109.6	3.17	113.39	3.1	111.1	7.6	276.0	7.8	282.0	7.8	283.5	7.844	285.477
47	3.0	112.6	3.23	116.56	3.1	114.2	7.6	283.5	7.8	289.8	7.9	291.4	7.987	293.464
48	3.1	115.7	3.30	119.79	3.2	117.4	7.8	291.3	8.0	297.8	8.1	299.5	8.138	301.602
49	3.2	118.9	3.36	123.09	3.2	120.6	8.1	299.4	8.0	305.8	8.3	307.8	8.297	309.899
50	3.2	122.1	3.43	126.45	3.3	123.9	8.1	307.4	8.2	314.0	8.4	316.2	8.466	318.365
51	3.0	125.1	3.51	129.88	3.4	127.3	7.6	315.0	8.2	322.2	8.6	324.8	8.644	327.009
52	3.3	128.4	3.59	133.39	3.5	130.8	8.3	323.3	8.6	330.8	8.8	333.6	8.832	335.841
53	3.4	131.8	3.67	136.98	3.5	134.3	8.6	331.9	8.9	339.7	9.0	342.6	9.032	344.873
54	3.6	135.4	3.76	140.65	3.6	137.9	9.1	340.9	9.0	348.7	9.2	351.8	9.243	354.116
55	3.6	139.0	3.86	144.41	3.7	141.6	9.1	350.0	9.4	358.1	9.4	361.2	9.468	363.584
56	3.6	142.6	3.96	148.26	3.8	145.4	9.1	359.1	9.7	367.8	9.7	370.9	9.707	373.291
57	3.7	146.3	4.06	152.22	3.9	149.3	9.3	368.4	9.9	377.7	9.9	380.8	9.962	383.253
58	3.8	150.1	4.18	156.29	4.0	153.3	9.6	378.0	10.4	388.1	10.2	391.0	10.233	393.486
59	4.1	154.2	4.30	160.46	4.1	157.4	10.3	388.3	10.5	398.6	10.5	401.5	10.523	404.009
60	4.1	158.3	4.43	164.77	4.3	161.7	10.3	398.6	10.7	409.3	10.8	412.3	10.833	414.842
61	4.1	162.4	4.58	169.20	4.4	166.1	10.3	408.9	11.0	420.3	11.1	423.4	11.166	426.008
62	4.4	166.8	4.73	173.78	4.5	170.6	11.1	420.0	11.4	431.7	11.5	434.9	11.523	437.531
63	4.7	171.5	4.89	178.51	4.7	175.3	11.8	431.8	11.7	443.4	11.9	446.8	11.908	449.438
64	4.6	176.1	5.07	183.40	4.9	180.2	11.6	443.4	12.0	455.4	12.3	459.1	12.323	461.761
65	5.3	181.4	5.27	188.48	5.0	185.2	13.3	456.8	12.4	467.8	12.7	471.8	12.772	474.533
66	5.4	186.8	5.48	193.74	5.2	190.4	13.6	470.4	13.0	480.8	13.2	485.0	13.259	487.792
67	5.4	192.2	5.71	199.22	5.4	195.8	13.6	484.0	13.9	494.7	13.8	498.8	13.790	501.582
68	5.5	197.7	5.96	204.93	5.7	201.5	13.8	497.8	14.1	508.8	14.3	513.1	14.369	515.951
69	5.6	203.3	6.24	210.89	5.9	207.4	14.1	511.9	14.9	523.7	15.0	528.1	15.003	530.954
70	5.8	209.1	6.54	217.12	6.2	213.6	14.6	526.5	16.0	539.7	15.7	543.8	15.702	546.656
71	5.8	214.9	6.88	223.66	6.5	220.1	14.6	541.1	16.5	556.2	16.4	560.2	16.474	563.129
72	6.7	221.6	7.26	230.55	6.8	226.9	16.9	558.0	17.2	573.4	17.3	577.5	17.331	580.460
73	7.4	229.0	7.69	237.81	7.2	234.1	18.6	576.6	18.7	592.1	18.3	595.8	18.288	598.748
74	8.0	237.0	8.17	245.50	7.7	241.8	20.1	596.8	19.6	611.7	19.3	615.1	19.363	618.111
75	8.4	245.4	8.72	253.68	8.1	249.9	21.2	617.9	20.7	632.4	20.6	635.7	20.580	638.691
76			9.36	262.40	8.7	258.6			22.5	654.9	22.0	657.7	21.966	660.657
77			10.09	271.76	9.3	267.9			23.7	678.6	23.6	681.3	23.561	684.218
78			10.96	281.85	10.1	278.0			26.1	704.7	25.4	706.7	25.414	709.633
79			11.99	292.81	10.9	288.9			27.8	732.5	27.6	734.3	27.593	737.226

Appendix III

The comparability of different (re)constructions

R = 315 [mm]

Rek[KN] : $vgBA(\varphi) = trunc(2 \cdot \sin(½ \cdot rad_m) \cdot R \cdot \sec(\varphi - ½) \cdot rad_m) \cdot 10)/10$

$Rek[KN]_{without\ mid\text{-}latitude}$: $vgBA(\varphi) = trunc(2 \cdot \sin(½ \cdot rad_m) \cdot R \cdot \sec(\varphi) \cdot rad_m) \cdot 10)/10$

$Rek[KN]_{rad}$: $vgBA(\varphi) = trunc(2 \cdot \sin(½ \cdot rad) \cdot R \cdot \sec(\varphi - ½) \cdot rad) \cdot 10)/10$

D'Hollander : $Dy = R \cdot \{\cos(\varphi \cdot rad) \cdot [\tan((\varphi+½) \cdot rad) - \tan((\varphi-½) \cdot rad)]\}$

Gaspar : $dy = R \cdot \{\cos((\varphi-½) \cdot rad) \cdot [\tan((\varphi+½) \cdot rad) - \tan((\varphi-½) \cdot rad)]\}$

Theorie : $th = R \cdot \{\ln[\tan(79 \cdot rad) + \sec(79 \cdot rad)]\}$

1 rheinischer Fuß = 313.85 [mm]

$Rek[KN]_{1rhF}$: $vgBA(\varphi) = trunc(2 \cdot \sin(½ \cdot rad_m) \cdot 1rhF \cdot \sec(\varphi-½) \cdot rad_m) \cdot 10)/10$

$rad_m = (22/7)/180$

rad $\pi/180$

Sum over all 79 vgBA:

Basel	=	732.5 [mm]	Δ				Δ
Rek[KN]	=	734.3	1.8	Theorie	=	737.23	4.73
$Rek[KN]_{rad}$	=	733.2	0.7				
$Rek[KN]_{1rhF}$	=	731.5	-1	D'Hollander	=	749.44	16.94
$Rek[KN]_{without\ mid\text{-}latitude}$	=	746.2	13.7	Gaspar	=	761.75	29.25

Appendix IV

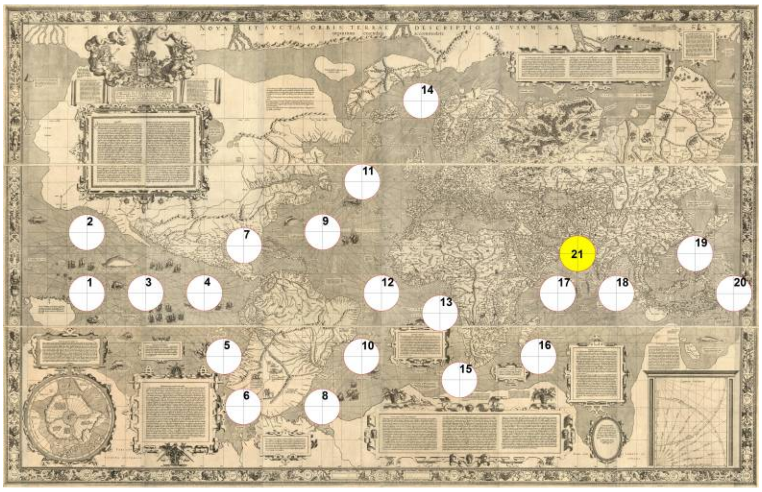
Various issues concerning Mercator's map 1569

I dedicate this appendix to the late Dr. Joseph Milz (†), former director of the Duisburg City Archives: In 1994 he allowed me to work with the 1 : 1 photos of the Basel copy of the original Mercator map.

Circles

At a first glance the „quadratura circuli” and the „loxodromy” are not related - as you must discuss the „quadratura circuli” in a special context: The paradigm shift from geometry to arithmetic and algebra. On the other hand, the circles which are to be squared become a subject of cartography as soon as you have to think about the influence of color, print, drying and aging of the paper used. Circles – particularly those in „old” maps - clearly show signs of distortion and shrinkage.

Fig. 19



Twenty-one rosettes on the map *Ad usum navigantium* are not distorted (after more than 430 years). [Basle copy 2009 / 2011 {Fak11} 600 dpi]

Fig. 20

The rosette number 21 cannot be recognized on the Fak94.

Quadrants

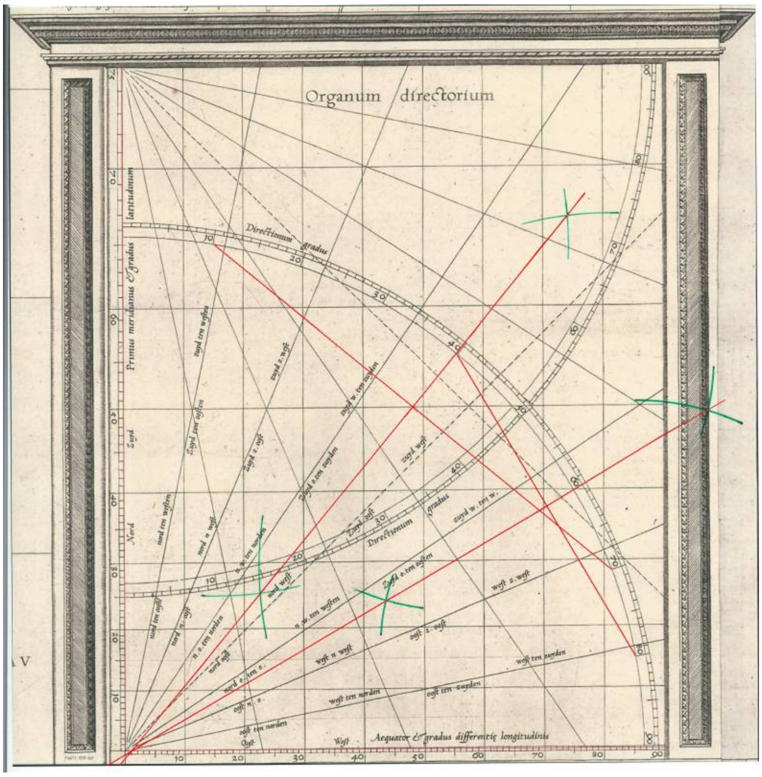
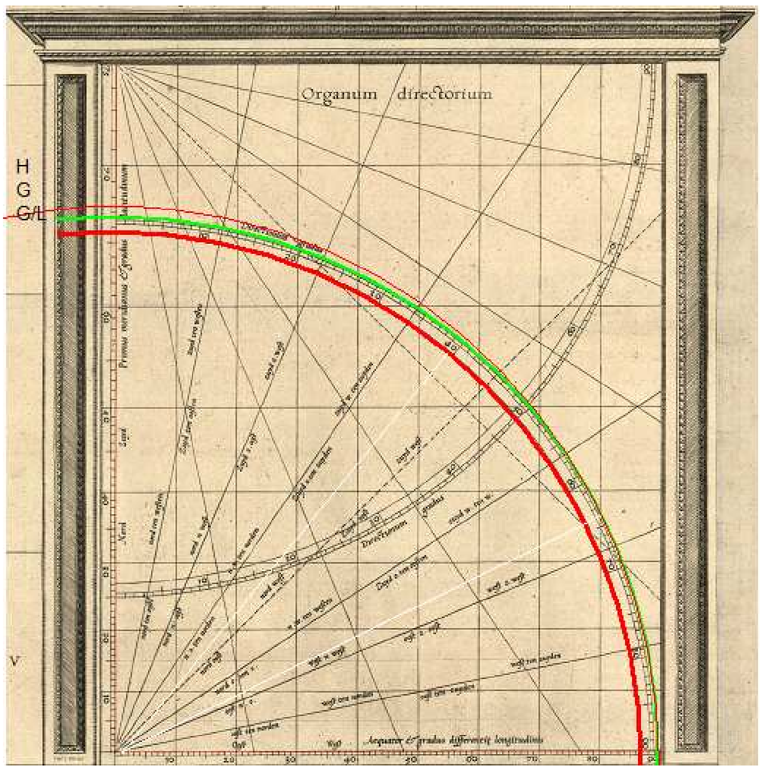


Fig. 21

1994: The construction of the centres of the quadrants presents (0°|1°) as well as (75°|1°).

Quarter circles or „quadrants” in old maps reveal much about the history of a cartographic map projection, about its fate and the way how the creator of a map has laid down his ideas finally. Taking this into account you will (easily) understand how it was possible not to discover a mistake in drawing in one quadrant of the map *Ad usum navigantium* for 425 years; the complete ignorance of this mistake lead to incorrect conclusions in the process of explaining the methodological construction of the map *Ad usum navigantium*.

Fig. 22



Correct quadrants have their center in (0°|1°) [as well as in (75°|1°)].
H, G, G/L:
A faulty construction [centered in (0°|0°)], that should justify a deformation of the organon - but there is no distortion at all:

Mercator drew the organon deliberately incorrectly.

All calculations in G, G/L use a quadrant-radius of R = 89.95°, „where R = 89.95[°] is the radius of the lower circle in equatorial degrees.” [13] But such a radius doesn't occur in the organon:

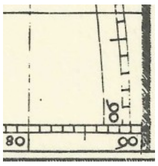
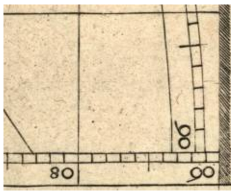


Fig. 23

Fig. 24

neither in the Fak94
nor in the Fak11:

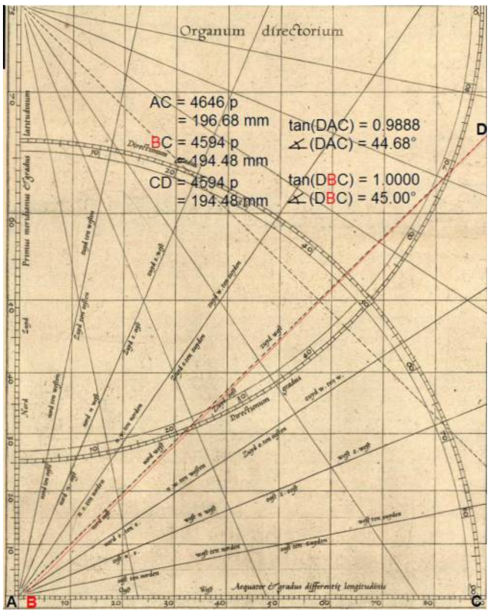


In reality, the radius of the outer quadrant only
amounts 89.95° - 1°; exact measurements (600 dpi) present 87.93° - start-point (0°|1°).

Also the scale of the quadrant in G/L must be reduced: 90° (G/L) => 89.3488°.

And this means: All - in this sense - not corrected calculations are faulty.[14]

Fig. 25



Fak11: Exact controls (600 dpi) confirm the result of 1994.

13 G/L 3 Fig. 1
14 For example: G/L 6, Table 2.

The Link

Raymond D'Hollander's beautiful book: *Loxodromie et projection de Mercator* (Paris 2005) criticizes all suggestions to solve the so-called „mystery of the Mercator map of 1569” and the author describes some of them as solutions „par quadrature”.

But the author of the *Loxodromie* ... - as well as those authors still following his ideas in the year of Mercator's 500th birthday - don't realize (sorry to say it) that the map projection *Ad usum navigantium* was - already before 1569 - linked by its creator with the age-old problem of „squaring the circle”, a link whose proof of an assumed connection was still missing in 1569.

One in the *Loxodromie* ... (and then by further authors 2012) rejected conjectures concerning the „mystery of the Mercator map of 1569” has lead to a proof of the connection assumed by Mercator himself – already in 1996.

A Concluding Conjecture

Gerhard Mercator possessed privileges for his map.[15] [16] But he feared unwarranted imitators:

> How simple would it be, to produce further maps on the basis
> of the simple and clear organon grid without any payment?

And it is not only meant ironically [17]:

> **In order to expose these imitators, he constructed the organon deliberately incorrectly –**
> **on the danger to be accused from the shrewd chairs,**
> **the controlling mathematicians and criticizing cartographers**
> **because of the wrong grid of the organon.**

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15 Ad usum navigantium 1569, Legend 15: *Cautum est privilegio*

16 Joseph Milz: *Das kaiserliche Privileg für die Weltkarte von 1569*, in Gerhard Mercator Weltkarte Ad usum navigantium Duisburg 1569 hrgg. von Wilhelm Krücken und Joseph Milz, Duisburg 1994, Beiheft 16 – 19.

17 gr. eironeia = to mock – from gr. eiron = one, that alters in his speech = the mischief, that places itself ignorantly
(The weapon, with which Socrates fought the sophists – as we know.)

Well - even in the 21st century, the alleged mistakes of Gerhard Mercator in the organum directorium - his error signature [18] - were documented only with the help of the deficient grid of the organum directorium.[19]

Why doesn't one examine the 350th longitude of the original map [for instance: ISBN 978-3-00-035705-3] of 1569?

Finis: Non opus ultra!

18 The summation of the restrictly small geometricaly-positioned increased latitude-sections [vgBA] are not in correct agreement with the summation of the infinite-small ones of the geometrical (Barrow) as well as the analytic calculus (Leibniz).

19 Honni soit qui mal y pense!